An Efficient Forecasting Procedure for Kuhn-Tucker Consumer Demand Model Systems

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ABSTRACT

This paper proposes an efficient and accurate forecasting algorithm for the MDCEV model. The algorithm builds on simple, yet insightful, analytical explorations with the Kuhn-Tucker conditions of optimality that shed new light on the properties of the MDCEV model. For specific, but reasonably general, functional forms of the consumption utility specification, the algorithm circumvents the need to carry out any iterative constrained optimization procedures that have hitherto been used for forecasting with Kuhn-Tucker (KT) demand model systems. The non-iterative nature of the algorithm contributes significantly to its efficiency and accuracy. Further, although developed in the context of the MDCEV model, the proposed algorithm can be easily modified to be used in the context of other utility maximization-based Kuhn-Tucker (KT) consumer demand model systems in the literature.

Simulation experiments highlight the efficiency of the algorithm compared to a traditional iterative forecasting procedure. For example, to forecast the expenditures of 4000 households in 7 expenditure alternatives, for 500 sets of error term draws for each household, the proposed algorithm takes less than 2 minutes. On the other hand, the iterative forecasting routine would take around 2 days to do so.

1. INTRODUCTION

In several consumer demand situations, consumer behavior may be associated with the choice of multiple alternatives simultaneously, along with a continuous choice component of "how much to consume" for the chosen alternatives. Such multiple discrete-continuous choice situations are being increasingly recognized and modeled in the recent literature in transportation, marketing, and economics.

A variety of modeling frameworks have been used to analyze multiple discrete-continuous choice situations. These can be broadly classified into: (1) statistically stitched multivariate single discrete-continuous models (see, for example, Srinivasan and Bhat, 2006), and (2) utility maximization-based Kuhn-Tucker (KT) demand systems (Hanemann, 1978; Wales and Woodland, 1983; Kim *et al.*, 2002; von Haefen and Phaneuf, 2005; Bhat, 2005 and 2008). Between the two approaches, the KT demand systems are more theoretically grounded in that they employ a unified utility maximization framework for simultaneously analyzing the multiple discrete and continuous choices. Further, these model systems accommodate fundamental features of consumer behavior such as satiation effects through diminishing marginal utility with increasing consumption.

The KT demand systems have been known for quite some time, dating back at least to the research works of Hanemann (1978) and Wales and Woodland (1983). However, it is only in the past decade that practical formulations of the KT demand system have appeared in the literature. Recent applications include, but are not limited to, individual activity participation and time-use studies (Bhat, 2005; Habib and Miller, 2009; Pinjari *et al.*, 2008), household travel expenditure analyses (Rajagopalan and Srinivasan, 2008; Ferdous *et al.*, 2009), household vehicle ownership and usage forecasting (Ahn *et al.*, 2007; Fang, 2008; and Bhat *et al.*, 2008), outdoor recreational demand studies (von Haefen *et al.*, 2004; and von Haefen and Phaneuf, 2005), and grocery purchase analyses (Kim *et al.*, 2002). As indicated by Vasqez-Lavín and Hanemann (2009), this surge in interest may be attributed to the strong theoretical basis combined with the recent developments in simulation techniques and econometric specifications that obviate the need for simulation-based estimation.

Within the KT demand systems, the recently formulated multiple discrete-continuous extreme value (MDCEV) model structure by Bhat (2005, 2008) is particularly attractive due to:

(1) its elegant closed-form probability expressions that simplify to the well-known multinomial logit probabilities when each decision-maker chooses only one alternative (Bhat, 2005), and
(2) its functional form of utility specification that enables a clear interpretation of the utility parameters and a convenient specification of the alternative attributes while maintaining the property of weak complimentarity (Bhat, 2008). The MDCEV model has been applied to analyze several of the choice situations identified earlier. Further, in recent papers, the basic MDCEV framework has been expanded in several directions, including the incorporation of more general error structures to allow flexible inter-alternative substitution patterns (Pinjari and Bhat, 2009; Pinjari, 2009).

Despite the many developments and applications, a simple and practically feasible forecasting procedure has not yet been developed for the MDCEV and other KT demand model systems. This has severely limited the applicability of these models for practical forecasting and other policy analysis purposes. Since the end-goal of model development and estimation is generally forecasting and policy evaluation, development of a simple and easily applicable forecasting procedure is a critical need in the area of KT demand model systems. This is not to say that no forecasting procedures exist in the literature (see Section 2). However, the currently available methods are either enumerative or iterative in nature, are not very accurate, and require large computation times.

In the context of the above discussion, this paper develops an efficient, non-iterative forecasting algorithm for the MDCEV model, which can also be applied for other KT demand model systems in the literature. The algorithm builds on simple, yet insightful, analytical explorations with the KT conditions that shed new light on the properties of the MDCEV model. For specific (but reasonably general) forms of utility functions, the algorithm results in analytically expressible consumptions obviating the need for approximation and minimizing the room for inaccuracy. Even with more general utility functions, the properties of the MDCEV model discovered in this paper can be used in designing efficient (albeit iterative) forecasting algorithms.

The remainder of the paper is organized as follows. The next section highlights the nature of the MDCEV forecasting problem and describes the currently used forecasting procedures in the literature. Section 3 discusses some new properties of the MDCEV model. Building on these properties, Section 4 presents the forecasting algorithm and some application results, along with

a discussion on how similar forecasting algorithms can be developed for other KT demand system models. Section 5 concludes the paper.

2 FORECASTING WITH KT DEMAND MODEL SYSTEMS

The MDCEV and other KT demand modeling systems are based on a resource allocation formulation. Specifically, it is assumed that the decision makers operate with a finite amount of available resources (*i.e.*, a budget), such as time or money. The decision-making mechanism is assumed to be driven by an allocation of the limited amount of resources to consume various goods/alternatives in such a way as to maximize the utility derived out of consumption. Further, a stochastic utility framework is used to recognize the analyst's lack of awareness of all factors affecting consumer decisions. In addition, a non-linear utility function is employed to incorporate important features of consumer choice making, including: (1) the diminishing nature of marginal utility with increasing consumption, and (2) the possibility of consuming multiple goods/alternatives as opposed to a single good/alternative. To summarize, the KT demand modeling frameworks are based on a stochastic (due to stochastic utility framework), constrained (due to the budget constraint), non-linear utility optimization formulation.

In most KT demand system models, the stochastic KT first order conditions of optimality form the basis for model estimation. Specifically, an assumption that stochasticity (or unobserved heterogeneity) is generalized extreme value (GEV) distributed leads to closed form consumption probability expressions (Bhat 2005 and 2008; Pinjari 2009; von Haefen *et al.*, 2004) facilitating a straightforward maximum likelihood estimation of the model parameters.

Once the model parameters are estimated and given a budget amount for each decision-maker, any forecasting or policy analysis exercise involves solving the stochastic, constrained, non-linear utility maximization problem for optimal consumption quantities of the decision-makers. Unfortunately, there is no straight-forward analytical solution to this problem; a combination of simulation (to mimic the unobserved heterogeneity) and optimization (to solve the constrained non-linear optimization problem) methods needs to be employed. That is, the analyst must carry out constrained non-linear optimization to obtain the consumption forecasts at each simulated value of unobserved heterogeneity (or stochasticity). Such conditional (on unobserved heterogeneity) consumption forecasts evaluated over the entire simulated distribution

of unobserved heterogeneity are used to derive the distributions of unconditional consumption forecasts.

To solve the conditional (on unobserved heterogeneity) constrained non-linear optimization problem, the forecasting procedures used in the literature so far use either enumerative or iterative optimization methods. The enumerative approach (used by Phaneuf *et al.*, 2000) involves enumeration of all possible sets of alternatives that the decision-maker can potentially choose from the available set of alternatives. Specifically, if there are K available choice alternatives, assuming not more than one essential Hicksian composite good (or outside good)[1], one can enumerate 2^K-1 possible choice set solutions to the consumer's utility maximization problem. Clearly, such a brute-force method becomes computationally burdensome and impractical even with a modest number of available choice alternatives/goods. Thus, for medium to large number of choice alternatives, iterative optimization procedures are the only available alternative approach till date.

The iterative optimization procedures, as with any iterative procedure, begin with an initial solution (for consumptions) that is improved in the subsequent steps (or iterations) by moving along specific directions using the gradients of the utility functions, until a desired level of accuracy is reached. Most studies in the literature use off-the-shelf optimization programs (such as the constrained maximum likelihood library of GAUSS) to carry out such iterative optimization. However, the authors' experience with iterative methods of forecasting in prior research efforts indicates several problems, including large computation time and convergence issues.

More recently, von Haefen *et al. (*2004) proposed a more efficient forecasting algorithm designed based on the insight that the optimal consumptions of all goods can be derived if the optimal consumption of the outside good is known. Specifically, conditional on the simulated values of unobserved heterogeneity, they iteratively solve for the optimal consumption of the outside good (as well as that of other goods) using a numerical bisection procedure until a desired level of accuracy is reached. The numerical bisection approach, although more efficient than a generic optimization program, is again an iterative procedure. As indicated earlier, using iterative optimization approaches for forecasting can be very time consuming, especially when

[1]Most KT demand system models generally include an essential Hiscksian good (or outside good, or numeraire good) which is always consumed by the decision-makers.

optimization is performed over the entire simulated distribution of unobserved heterogeneity. Hence, von Haefen *et al. (*2004) circumvent the need to perform predictions over the entire simulated distribution of unobserved heterogeneity by conditioning on the observed choices.[2] Based on Monte Carlo experiments with low-dimensional choice sets, they indicate that, relative to the unconditional approach (of simulating the entire distribution of unobserved heterogeneity), the conditional approach required about 1/3rd the simulations (of conditional unobserved heterogeneity) and time to produce stable estimates of mean consumptions and welfare measures. However, even such a conditional approach (of using the observed choice information) to forecasting (and welfare measurement) may benefit from having a non-iterative optimization procedure at the core. Further, in many situations, the model needs to be applied to data outside the estimation sample, and observed choices are not available. This is especially the case in the travel demand field, where models are estimated with an express intent to apply them for predicting the activity-travel patterns in the external (to estimation sample) data representing the study area population.[3]

3 THE MDCEV MODEL: STRUCTURE AND PROPERTIES

In this section, we draw from Bhat (2008) to briefly discuss the structure of the MDCEV model (Section 3.1), and then derive some fundamental properties of the model (Section 3.2) that will form the basis for the development of the forecasting algorithm.

3.1 Model Structure

Consider the following additively separable utility function as in Bhat (2008):

$$U(\boldsymbol{t}) = \frac{1}{\alpha_1}\psi_1 t_1^{\alpha_1} + \sum_{k=2}^{K} \frac{\gamma_k}{\alpha_k}\psi_k \left\{\left(\frac{t_k}{\gamma_k}+1\right)^{\alpha_k} - 1\right\}; \quad \psi_k > 0,\ 0 < \alpha_k \le 1,\ \gamma_k > 0 \tag{1}$$

[2]To do so, they simulate the unobserved heterogeneity that corresponds to the observed choices in the data. Using these simulated values of conditional (on observed choices) unobserved heterogeneity, they apply the numerical bisection algorithm to perform predictions for the policy case and subsequently compute the welfare change from the base case to policy case. Thus, by using the observed choices data and the conditional unobserved heterogeneity, they avoid the need for performing predictions for the base case and circumvent the need to perform forecasting over the entire distribution of unobserved heterogeneity. See von Haefen (2003) for a discussion on the advantages of incorporating observed choices into policy analyses involving random utility models.

[3]For example, Rajagopalan *et al. (*2009) and Pinjari and Bhat (2009) estimated MDCEV models of individual's time allocation to different activity types during different times-of-day. The end-goal of these models is to serve as daily activity generation modules (to predict the types and timing of activities undertaken by an individual in a day) in larger activity-based travel demand model systems that are intended to predict the activity-travel patterns in the entire study area population for transportation planning and policy making purposes.

In the above expression, $U(\boldsymbol{t})$ is the total utility accrued from consuming $\boldsymbol{t}$ (a Kx1 vector with non-negative consumption quantities t_k; k = 1,2,…,K) amount of the K alternatives available to the decision maker. The ψ_k terms (k = 1,2,…,K), called as the baseline utility parameters, represents the marginal utility of one unit of consumption of alternative k at the point of zero consumption for that alternative (Bhat, 2008). Through the ψ_k terms, the impact of observed and unobserved alternative attributes, decision-maker attributes, and the choice environment attributes may be introduced as $\psi_k = \exp(\beta' z_k + \varepsilon_k)$, where z_k contains the observed attributes and ε_k captures the unobserved factors. The α_k terms (k = 1,2,…,K), labeled as satiation parameters $(0 < \alpha_k \leq 1)$, capture satiation effects by reducing the marginal utility accrued from each unit of additional consumption of alternative k (Bhat, 2008).[4] The γ_k terms (k = 2,3,…,K), labeled as translation parameters, play a similar role of satiation as that of α_k terms, and an additional role of translating the indifference curves associated with the utility function to allow corner solutions (*i.e.*, accommodate the possibility that decision-makers may not consume all alternatives; see Bhat, 2008). As it can be observed, there is no γ_k term for the first alternative for it is assumed to be an essential Hicksian composite good (or outside good or essential good) that is always consumed (hence no need for corner solution). Finally, the consumption-based utility function in (1) can be expressed in terms of expenditures (e_k) and prices (p_k) as:[5]

$$U(\boldsymbol{e}) = \frac{1}{\alpha_1}\psi_1\left(\frac{e_1}{p_1}\right)^{\alpha_1} + \sum_{k=2}^{K}\frac{\gamma_k}{\alpha_k}\psi_k\left\{\left(\frac{e_k}{\gamma_k p_k}+1\right)^{\alpha_k}-1\right\}, \text{ where } \frac{e_k}{p_k} = x_k \tag{2}$$

From the analyst's perspective, the decision-makers maximize the random utility given by Equation (2) subject to a linear budget constraint and non-negativity constraints on t_k:

$$\sum_{k=1}^{K} p_k x_k = E \text{ (where } E \text{ is the total budget) and } x_k \geq 0 \;\forall\; k \;(k = 1, 2, ..., K) \tag{3}$$

[4] Theoretically speaking, the α_k values can be negative. But, as indicated by Bhat (2008), imposing the condition $\alpha_k > 0$ provides much needed stability in empirical estimations.

[5] For the first alternative, $p_1 = 1$, since it is the "numeraire" good. However, in the exposition in the paper, we will use the notation p_1 rather than setting this to 1.

The optimal consumptions (or expenditure allocations) can be found by forming the Lagrangian and applying the Kuhn-Tucker (KT) conditions. The Lagrangian function for the problem is:

$$\mathcal{L} = \frac{1}{\alpha_1}\psi_1\left(\frac{e_1}{p_1}\right)^{\alpha_1} + \sum_{k=2}^{K}\frac{\gamma_k}{\alpha_k}\psi_k\left\{\left(\frac{e_k}{\gamma_k p_k}+1\right)^{\alpha_k}-1\right\} - \lambda\left[\sum_{k=1}^{K} e_k - E\right],$$

where λ is the Lagrangian multiplier associated with the budget constraint. The KT first-order conditions for the optimal expenditure allocations $(e_k^*; k = 1, 2, ..., K)$ are given by:

$$\frac{\psi_1}{p_1}\left(\frac{e_1^*}{p_1}\right)^{\alpha_1-1} - \lambda = 0, \text{ since } e_1^* > 0,$$

$$\frac{\psi_k}{p_k}\left(\frac{e_k^*}{\gamma_k p_k}+1\right)^{\alpha_k-1} - \lambda = 0, \text{ if } e_k^* > 0, \ (k = 2,\ldots, K) \tag{4}$$

$$\frac{\psi_k}{p_k}\left(\frac{e_k^*}{\gamma_k p_k}+1\right)^{\alpha_k-1} - \lambda < 0, \text{ if } e_k^* = 0, \ (k = 2,\ldots, K)$$

As indicated earlier, these stochastic KT conditions form the basis for model estimation (see Bhat, 2008). Next, using these same stochastic KT conditions, we derive a few properties of the MDCEV model that can be exploited to develop a highly efficient forecasting algorithm.

3.2 Model Properties

Property 1: *The price-normalized baseline utility of a chosen good is always greater than that of a good that is not chosen.*

$$\left(\frac{\psi_i}{p_i}\right) > \left(\frac{\psi_j}{p_j}\right) \text{ if 'i' is a chosen good and 'j' is not a chosen good.} \tag{5}$$

Proof: The KT conditions in (4) can be rewritten as:

$$\frac{\psi_1}{p_1}\left(\frac{e_1^*}{p_1}\right)^{\alpha_1-1} = \lambda,$$

$$\frac{\psi_k}{p_k}\left(\frac{e_k^*}{\gamma_k p_k}+1\right)^{\alpha_k-1} = \lambda, \text{ if } e_k^* > 0, \ (k = 2,\ldots, K) \text{ (\textit{i.e.}, for all chosen goods)} \tag{6}$$

$\frac{\psi_k}{p_k} < \lambda$, if $e_k^* = 0$, (k = 2,…, K) (*i.e.*, for all goods that are not chosen)

The above KT conditions can further be rewritten as:

$$\left(\frac{\psi_k}{p_k}\right) = \left(\frac{e_k^*}{\gamma_k p_k} + 1\right)^{1-\alpha_k} \left(\frac{\psi_1}{p_1}\right)\left(\frac{e_1^*}{p_1}\right)^{\alpha_1 - 1}, \quad \text{if } e_k^* > 0, (k = 2,3,...,K) \tag{7}$$

$$\left(\frac{\psi_k}{p_k}\right) < \left(\frac{\psi_1}{p_1}\right)\left(\frac{e_1^*}{p_1}\right)^{\alpha_1 - 1}, \quad \text{if } e_k^* = 0, (k = 2,3,...,K)$$

Now, consider two alternatives *'i'* and *'j'*, of which *'i'* is chosen and *'j'* is not chosen by a consumer. For that consumer, the above KT conditions for alternatives *'i'* and *'j'* can be written as:

$$\left(\frac{\psi_i}{p_i}\right) = \left(\frac{e_i^*}{\gamma_i p_i} + 1\right)^{1-\alpha_i} \left(\frac{\psi_1}{p_1}\right)\left(\frac{e_1^*}{p_1}\right)^{\alpha_1 - 1}, \text{ and} \tag{8}$$

$$\left(\frac{\psi_j}{p_j}\right) < \left(\frac{\psi_1}{p_1}\right)\left(\frac{e_1^*}{p_1}\right)^{\alpha_1 - 1}$$

Further, since $\left(\frac{e_j^*}{\gamma_j p_j} + 1\right)^{1-\alpha_j}$ is always greater than 1, one can write the following inequality:

$$\left(\frac{\psi_j}{p_j}\right) < \left(\frac{\psi_1}{p_1}\right)\left(\frac{e_1^*}{p_1}\right)^{\alpha_1 - 1} < \left(\frac{e_i^*}{\gamma_i p_i} + 1\right)^{1-\alpha_i} \left(\frac{\psi_1}{p_1}\right)\left(\frac{e_1^*}{p_1}\right)^{\alpha_1 - 1} \tag{9}$$

As one can observe, the third term in the above inequality is nothing but $\left(\frac{\psi_i}{p_i}\right)$. Thus, by the transitive property of inequality of real numbers, the above inequality implies a fundamental property of the MDCEV model that $\left(\frac{\psi_i}{p_i}\right) > \left(\frac{\psi_j}{p_j}\right)$. In words, the price-normalized baseline utility of a chosen good is always greater than that of a good that is not chosen.

Corollary 1.1: *It naturally follows from the property above that when all the K alternatives/goods available to a consumer are arranged in the descending order of their price-*

normalized baseline utility values (with the outside good being the first in the order), and if it is known that the number of chosen alternatives is M, then one can easily identify the chosen alternatives as the first M alternatives in the arrangement.[6]

Property 2: *The minimum consumption amount of the outside good is* $\left(\frac{\left(\frac{\psi_1}{p_1}\right)}{\max\limits_{\forall k=(2,3,...,K)}\left(\frac{\psi_k}{p_k}\right)} \right)^{\frac{1}{1-\alpha_1}}$.

Proof: Using the first and third KT conditions in (6), and considering market baskets that involve only the consumption of the outside good (*i.e.,* $e_k^* = 0, \forall k > 1$). At these market baskets, one can write the following:

$$\left(\frac{\psi_k}{p_k}\right) < \frac{\psi_1}{p_1}\left(\frac{e_1^*}{p_1}\right)^{\alpha_1 - 1} \; ; \; \forall k = (2,3,...,K),$$

or, $\max\limits_{\forall k=(2,3,...,K)}\left(\frac{\psi_k}{p_k}\right) < \frac{\psi_1}{p_1}\left(\frac{e_1^*}{p_1}\right)^{\alpha_1 - 1}$.

$$\text{Hence, } \left(\frac{e_1^*}{p_1}\right) > \left(\frac{\left(\frac{\psi_1}{p_1}\right)}{\max\limits_{\forall k=(2,3,...,K)}\left(\frac{\psi_k}{p_k}\right)} \right)^{\frac{1}{1-\alpha_1}} . \qquad (10)$$

In words, the right side of the above equation represents the "minimum" amount of consumption of the outside good. The interpretation is that, after the "minimum" amount of outside good is consumed, all other goods (and the outside good) start competing for the remaining amount of the budget. Thus, if the budget amount is less than that corresponding to the minimum consumption of outside good given in (10), no other goods will be consumed. Another interesting point to note here is that if there is no price variation across the consumption alternatives, the distribution of the minimum consumption of the outside good can be derived as a log-logistic variable.

[6]The reader is cautioned to note here that the converse of this property may not always hold true. That is, given the price-normalized utilities of two alternatives, one can not say with certanity if one (with higher value of price-normalized utility) or both of the alternatives are chosen.

Property 3: *When all the satiation parameters* (α_k) *are equal, and if the corner solutions are known (i.e., if the chosen and not-chosen alternatives are known), the Lagrange multiplier of the utility maximization as well as the optimal consumptions of the chosen goods can be expressed in an analytical form.*

Proof:[7] Using the first and second KT conditions in (6), and assuming without loss of generality that the first *M* goods are chosen, one can express the optimal consumptions as:

$$\begin{aligned} &\frac{e_1^*}{p_1} = \left(\lambda \frac{p_1}{\psi_1}\right)^{\frac{1}{\alpha_1 - 1}}, \text{ and} \\ &\frac{e_k^*}{p_k} = \left\{\left(\lambda \frac{p_k}{\psi_k}\right)^{\frac{1}{\alpha_k - 1}} - 1\right\}\gamma_k; \ \forall k = (2,3,...,M) \end{aligned} \tag{11}$$

Using these expressions, the budget constraint in (3) can be written as:

$$p_1\left(\lambda \frac{p_1}{\psi_1}\right)^{\frac{1}{\alpha_1 - 1}} + \sum_{k=2}^{M} p_k \left\{\left(\lambda \frac{p_k}{\psi_k}\right)^{\frac{1}{\alpha_k - 1}} - 1\right\}\gamma_k = E$$

From the above equation, and assuming that all satiation (α_k) parameters as equal to α, the Lagrange multiplier λ can be expressed analytically as:

$$\lambda = \left(\frac{E + \sum_{k=2}^{M} p_k \gamma_k}{p_1\left(\frac{\psi_1}{p_1}\right)^{\frac{1}{1-\alpha}} + \sum_{k=2}^{M} p_k \gamma_k \left(\frac{\psi_k}{p_k}\right)^{\frac{1}{1-\alpha}}}\right)^{\alpha - 1} \tag{12}$$[8]

The above expression for λ can be substituted back into the expressions in (11) to obtain the following analytical expressions for optimal consumptions:

[7] To be sure, this is a standard property and has been proved elsewhere, albeit with different functional forms. Nonetheless, we provide a proof here for completeness.

[8] The reader will note here that this expression contains terms corresponding to the consumed goods only.

$$\frac{e_1^*}{p_1} = \frac{\left(\frac{\psi_1}{p_1}\right)^{\frac{1}{1-\alpha}}\left(E+\sum_{k=2}^{M} p_k\gamma_k\right)}{p_1\left(\frac{\psi_1}{p_1}\right)^{\frac{1}{1-\alpha}} + \sum_{k=2}^{M} p_k\gamma_k\left(\frac{\psi_k}{p_k}\right)^{\frac{1}{1-\alpha}}} \tag{13}$$

$$\frac{e_k^*}{p_k} = \left(\frac{\left(\frac{\psi_k}{p_k}\right)^{\frac{1}{1-\alpha}}\left(E+\sum_{k=2}^{M} p_k\gamma_k\right)}{p_1\left(\frac{\psi_1}{p_1}\right)^{\frac{1}{1-\alpha}} + \sum_{k=2}^{M} p_k\gamma_k\left(\frac{\psi_k}{p_k}\right)^{\frac{1}{1-\alpha}}} - 1\right)\gamma_k;\ \forall k = (2,3,...,M). \tag{14}$$

4 AN EFFICIENT FORECASTING ALGORITHM

In this section, using the properties identified in the preceding section, we develop an efficient, non-iterative forecasting algorithm for the MDCEV model with the following utility functional form:

$$U(\boldsymbol{e}) = \frac{1}{\alpha}\exp(\beta' z_1 + \varepsilon_1)\left(\frac{e_1}{p_1}\right)^{\alpha} + \sum_{k=2}^{K} \frac{\gamma_k}{\alpha}\exp(\beta' z_k + \varepsilon_k)\left\{\left(\frac{e_k}{\gamma_k p_k}+1\right)^{\alpha} - 1\right\}, \tag{15}$$

As indicated in Bhat (2008) this is a γ-profile utility function in that the γ_k parameters are different across the choice alternatives but the satiation (α_k) parameters are constrained to be equal (to α) across all choice alternatives. Further, as $\alpha \to 0$, this utility form converges to a Linear Expenditures System (LES) as in Hanemann (1978).

4.1 Outline of the Forecasting Algorithm

The proposed algorithm comprises of four basic steps as outlined below.

Begin Algorithm

Step 0: Assume that only the outside good is chosen and let the number of chosen goods *M* = 1.

Step 1: Given the input data (z_k, p_k), model parameters (β, γ_k, α) and the simulated error term (ε_k) draws, compute the price-normalized baseline utility values $\left(\exp(\beta' z_k + \varepsilon_k)/p_k\right)$ for all alternatives. Arrange all the *K* alternatives available to the consumer in the descending order of their price-normalized baseline utility values (with the outside good in the first place). Go to step 2.

Step 2: Compute the value of λ using equation (12). Go to step 3.

Step 3: If (λ > the price-normalized baseline utility of the alternative in position *M+1*)

Compute the optimal consumptions of the first *M* alternatives in the above descending order using equations in (13) and (14). Set the consumptions of other alternatives as zero and stop.

Else, go to step 4.

Step 4: *M = M+1*.

If (*M = K*)

Compute the optimal consumptions using equations in (13) and (14) and stop.

Else, go to step 2.

End Algorithm.

4.2 Application of the Algorithm

The above-outlined algorithm can be applied a large number of times with different simulated values of the ε_k terms to sufficiently cover the simulated distribution of unobserved heterogeneity (*i.e.*, the ε_k terms) and obtain the distributions of the consumption forecasts. Alternatively, if the observed consumption choices are available, as in von Haefen *et al.* (2004), to perform policy analyses, one can apply the algorithm with simulated values of conditional (on observed choices) unobserved heterogeneity. The procedure for simulating the conditional unobserved heterogeneity is discussed below.

The KT conditions in (4), after taking logarithms and algebraic arrangements, can be written as follows:

$$\varepsilon_k - \varepsilon_1 = V_1 - V_k \text{ if } e_k^* > 0, \ (k = 2, 3, \ldots, K)$$

$$\varepsilon_k - \varepsilon_1 < V_1 - V_k \text{ if } e_k^* = 0, \ (k = 2, 3, \ldots, K) \qquad (16)$$

where, $V_1 = \beta' z_1 + (\alpha_1 - 1)\ln\left(\frac{e_1^*}{p_1}\right) - \ln p_1$, and $V_k = \beta' z_k + (\alpha_k - 1)\ln\left(\frac{e_k^*}{\gamma_k p_k} + 1\right) - \ln p_k$.

One can observe from the above equations that infinite combinations of the values of ε_1 and ε_k ($k = 2,3,\ldots,K$) terms result in the same KT conditions (and consumptions) as long as the difference between ε_k and ε_1 remains the same. That is, only the differences in error terms

matter (Bhat, 2008). Based on this insight, one can recast the MDCEV model with a differenced-error structure where the outside good is associated with no error term and the remaining *K-1* goods are associated with a multivariate logistic distributed error structure (*i.e.*, $\tilde{\varepsilon}_{k,1} = \varepsilon_k - \varepsilon_1;\ k = 2,3,...,K$; see Appendix C of Bhat, 2008 for more details). Thus, it is sufficient to simulate the differences in error terms ($\tilde{\varepsilon}_{k,1} = \varepsilon_k - \varepsilon_1$) rather than the error terms (ε_k and ε_1) themselves. Based on this insight, one can assume that the differenced-error term for the outside good is zero (*i.e.*, $\tilde{\varepsilon}_{1,1} = \varepsilon_1 - \varepsilon_1 = 0$) and simulate the differenced-unobserved error terms for other alternatives. For chosen alternatives, one can easily do so by using the first equation in (16) (*i.e.*, $\tilde{\varepsilon}_{k,1} = V_k - V_1$). To simulate the differenced-error terms for the not-chosen alternatives, the analyst has to draw from a truncated multivariate logistic distribution resulting from the second equation in (16).[9]

4.3 Intuitive Interpretation of the Algorithm

As it can be observed, the proposed algorithm in Section 4.1 builds on the insight from corollary 1.1 that if the number of chosen alternatives is known, one can easily identify the chosen alternatives by arranging the price-normalized baseline utility values in a descending order. Subsequently, one can compute the optimal consumptions of the chosen alternatives using Equations (13) and (14). The only issue, however, is that the number of chosen alternatives is unknown *apriori*. To find this out, the algorithm begins with an assumption that only one alternative (*i.e.*, the outside good) is chosen and verifies this assumption by verifying the KT conditions (*i.e.*, the condition in Step 3) for other, (assumed to be)not-chosen goods. If the KT conditions (*i.e.*, the condition in Step 3) are met, the algorithm stops. Else, at least the next alternative (in the order of the price-normalized baseline utilities) has to be among the chosen alternatives. Thus, the KT conditions (*i.e.*, the condition in step 3) are verified again by assuming that the next alternative is among the chosen alternatives. These basic steps are repeated until either the KT conditions (*i.e.*, the condition in step 3) are met or the assumed number of chosen alternatives reaches the maximum number (*K*).

[9] The reader will note here that the procedure to simulate the conditional (on observed choices) error terms described here for the MDCEV model is different from that in von Haefen *et al.* (2004), due to the subtle (but important) differences in the assumptions on stochasticity between the MDCEV and von Haefen *et al.* models (see Bhat, 2008).

As one may note from the above description, the KT conditions are essentially replaced by a single condition (involving the Lagrange multiplier λ) in Step 3 of the algorithm. This condition is equivalent to verifying if λ is greater than the highest price-normalized baseline utility among the not-chosen goods. To understand this better, recall from the equations in (6) that verifying the KT conditions in this algorithm is nothing but verifying the condition $\lambda > \frac{\psi_k}{p_k}$ for all goods that are assumed to have not been chosen. Obviously, verifying the condition in Step 3 (*i.e.*, if λ is greater than the highest price-normalized baseline utility of all goods assumed to have not been chosen) is the most efficient way of doing so.[10]

The proposed algorithm involves enumeration of the choice baskets in the most efficient fashion. In fact, the algorithm begins with identifying a single alternative (outside good) that may be chosen. If the KT conditions are not met for this choice basket, the algorithm identifies a two-alternative choice basket and so on, till the number of chosen alternatives is determined. Thus, the number of times the algorithm enumerates choice baskets is equal to the number of chosen alternatives in the optimal consumption portfolio, which is at most equal to (but many times less than) the total number of available alternatives (*K*).

Another feature of the algorithm is that it is not iterative in nature, which makes it highly efficient compared to other, iterative approaches. Further, coding the algorithm using vector and matrix notation in matrix programming languages significantly reduces the computational burden even with large number of choice alternatives and observations. Also, due to the convenient γ-profile utility specification, the algorithm is accurate with no room for any inaccuracy (unlike the existing iterative procedures discussed earlier) as it uses analytical expressions for the optimal consumption computations.

In summary, the proposed algorithm is simple, efficient, and accurate. The only disadvantage of this algorithm (in its current form) is it is designed to be used with the γ-profile utility specification. Admittedly, the γ-profile utility specification is not the most general form within the class of additively separable utility functions. However, as indicated in Bhat (2008), both the γ_k and α_k parameters serve the role of allowing differential satiation effects across the choice alternatives. Due to the overlapping roles played by these parameters, attempts to estimate

[10] At the beginning of the algorithm, when only the outside good is assumed to be consumed, the condition in Step 3 of the algorithm is equivalent to the "minimum consumption" condition in Equation (7).

utility functions that allow both the γ_k and α_k parameters to vary across alternatives may lead to severe empirical identification issues and estimation breakdowns. Further, "*for a given ψ_k value, it is possible to closely approximate a sub-utility function based on a combination of γ_k and α_k values with a sub-utility function solely based on γ_k or α_k values*" (Bhat, 2008). For these reasons, and given the ease of forecasting with the proposed algorithm (see Section 4.4), we suggest an estimation of the γ-profile utility function. Nevertheless, the insights obtained from the properties 1 and 2 discussed in the preceding section (and parts of this algorithm) can be used to design an efficient (albeit iterative) algorithm for cases when α_k parameters vary across alternatives. For example, the numerical bisection algorithm of von Haefen *et al.*, which can be used for forecasting with the MDCEV model when α_k parameters vary, can potentially be made more efficient based on the insights obtained from properties 1 and 2.

4.4 Generalization of the Forecasting Algorithm for other KT Demand Model Systems

The algorithm developed in Section 4.1 is tailored to the MDCEV model. However, similar properties as in Section 3 can be derived, and similar forecasting algorithms can be developed for other KT demand model systems in the literature. Specifically, the basic algorithmic approach remains the same as that in Section 4.1, except that steps 1, 2, and 3 of the algorithm vary slightly depending on the form of the utility function employed in different KT demand systems. Specifically, the following modifications will be needed for a different form of utility function:

- In step 1, all the *K* available alternatives have to be arranged in the descending order based on a measure that depends on the form of the utility function (with the outside good being the first in the order). Such a measure can be easily derived for different forms of utility functions used in the literature by following the derivations of the proof of property 1 in Section 3.1. For the MDCEV model, as shown in Section 3.1, this measure is the price-normalized baseline utility. That is, for the MDCEV model, the alternatives have to be arranged in the descending order of their price-normalized baseline utility values.[11]

[11]This discussion suggests a general property of the KT demand system models used in the literature so far (with additive utility functions) that the choice alternatives can always be arranged in the descending order of a specific measure that depends on the form of the utility function. The generality of this property and the conditions under which such a measure may not exist is yet to be thoroughly investigated.

- In step 2, the formula for the computation of the Lagrange multiplier (λ) will depend on the form of the utility function. The analytical formula for λ can be derived following the proof of property 3 in Section 3.1, as long as the α_k parameters do not vary across different alternatives.
- In step 3, the Lagrange multiplier (λ) will have to be compared with the same measure that was used to arrange the *K* alternatives in the descending order. The formulae for the optimal consumptions (conditional on the knowledge of the consumed alternatives) will also depend on the form of the utility function. Again, these formulae can also be derived in a straight forward fashion for utility functions with same α_k parameter across all alternatives.

Thus, the proposed algorithm is fairly generic in its approach, and can be easily modified to be used with different forms of utility functions.

4.5 Application Results

Limited experiments were conducted to assess the performance of the algorithm. Specifically, the performance of the proposed algorithm and that of a traditionally used iterative optimization routine were compared. Empirical data on household transportation expenditures, obtained from the 2002 Consumer Expenditure Survey conducted by the Bureau of Labor Statistics (BLS, 2003), available at the National Bureau of Economic Research data archives (NBER, 2003), was used for the experiments. This data was extracted by Ferdous *et al. (*2009) and used in Pinjari (2009) to analyze household expenditures in six transportation categories (or alternatives), including: (1) Vehicle purchases, (2) Gasoline and motor oil, (3) Vehicle insurance, (4) Vehicle maintenance, (5) Air travel, and (6) Public transportation, and a seventh, outside category that includes all other expenditures of a household. An MDCEV model with these seven expenditure alternatives was estimated using data from 4000 households. These model parameters and the data of the 4000 households were used for subsequent experiments.

The proposed algorithm was coded and executed in Gauss matrix programming language. In addition, for comparison purposes, the iterative constrained optimization routines of the Constrained Maximum Likelihood (CML) module of Gauss were also used for forecasting the household expenditures in the empirical data. The CML module of Gauss uses a sequential programming method for carrying out non-linear optimization, in which the optimal consumption values are approximated iteratively using the first and second gradients of the

Lagrangian function. To recognize stochasticity, both the forecasting procedures were run repeatedly using several sets of Halton draws of the ε_k terms.

To forecast the expenditure patterns of 4000 households in the seven expenditure alternatives identified above, the proposed algorithm takes less than 2 minutes with 500 sets of error terms draws for each household. On the other hand, the optimization routine in the (CML) module of Gauss takes at least 6 minutes to compute the expenditure patterns of the same 4000 households for just one set of error term draws for each household. A linear extrapolation to 500 sets of error term draws implies a rather large computation time of more than 2 days. These run time differences (2 minutes versus 2 days) clearly highlight the efficiency of the proposed algorithm. Even in empirical contexts with a large number of alternatives, observations, or error term draws, the computation time of the proposed algorithm will not increase in a linear fashion. This is because, since most of the algorithm can be executed using matrix operations, the computations are performed simultaneously for all observations and over all error term draws. This contributes to the significant computational efficiency of the proposed algorithm. On the other hand, the iterative forecasting procedure does not exhibit such computational efficiencies.

In addition to being computationally efficient, the proposed algorithm was found to be more accurate than the iterative procedure. Specifically, minor differences were found between the predicted expenditures of the two procedures for a small number of cases. In such cases, the utility value corresponding to the predictions from the proposed algorithm was higher compared to that of the iterative procedure, indicating the relative inaccuracy of the iterative procedure. In this context, the proposed algorithm is analytical in nature and hence does not leave room for any approximations or inaccuracy. Finally, in certain (although a small number of) instances, the iterative procedure ran into convergence problems, and either yielded suboptimal solutions or did not even converge. On the other hand, the proposed algorithm did not run into any convergence issues thanks to its non-iterative nature.

5 SUMMARY AND CONCLUSIONS

This paper proposes a simple, elegant, efficient, and accurate forecasting algorithm for the MDCEV model. The algorithm builds on simple, yet insightful, analytical explorations with the Kuhn-Tucker conditions of optimality that shed new light on the properties of the model. For specific, but reasonably general, forms of consumption utility functions, the algorithm

circumvents the need to carry out any iterative constrained optimization procedures that have hitherto been used for forecasting with Kuhn-Tucker (KT) demand system models. The non-iterative nature of the algorithm contributes significantly to its efficiency and accuracy. Further, although developed in the context of the MDCEV model, the proposed algorithm can be easily modified to be used in the context of other utility maximization-based Kuhn-Tucker (KT) consumer demand model systems in the literature.

Results from the application of the proposed algorithm and a traditionally used iterative algorithm indicate the significant computational efficiency (as well as accuracy) of the proposed algorithm. The appealing feature of the algorithm is that most of it can be in executed using matrix operations, which enable a simultaneous computation for a large number of observations and huge computational efficiencies.

As the proposed algorithm makes it easier to perform forecasting and policy analysis with MDCEV and other KT demand model systems, it is hoped that these types of models will soon be utilized for practical transportation planning and policy analysis purposes. Further, now that a fast and easy-to-use forecasting procedure is available, further research can be conducted to characterize the distributions of forecasts obtained from MDCEV and other KT demand model systems.

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